Linear Programming

Definitions and Properties (cont.)

An LP is in the **augmented form** if all the variables are nonnegative and all the functional constraints are equations (built with **slack/surplus variables**, if necessary).

An **augmented solution** is a solution for an LP problem in the augmented form.

Property 5: Every LP problem can be rewritten as an equivalent problem in the maximisation augmented form.

Proof (writing an LP problem as a maximisation augmented form problem)

(i) $Min\ Z = \sum_{j=1}^{n} c_j x_j$ ➔ $Max(-Z) = \sum_{j=1}^{n} (-c_j)\, x_j$ $\quad (Min\ Z = -Max(-Z))$

(ii) '$\leq$' constraint: $\sum_{j=1}^{n} a_{ij}\, x_j \leq b_i \Leftrightarrow \sum_{j=1}^{n} a_{ij}\, x_j + x_{n+i} = b_i \ \wedge\ x_{n+i} \geq 0$

(iii) '$\geq$' constraint: $\sum_{j=1}^{n} a_{kj}\, x_j \geq b_k \Leftrightarrow \sum_{j=1}^{n} a_{kj}\, x_j - x_{n+k} = b_k \ \wedge\ x_{n+k} \geq 0$

(iv) nonpositive variable: $x_j \leq 0 \Leftrightarrow x'_j = -x_j \geq 0$

(v) free variable: x_j free $\Rightarrow x_j = x'_j - x''_j$, with $\begin{cases} x'_j = Max\{0 ; x_j\} \geq 0 \\ x''_j = Max\{0; -x_j\} \geq 0 \end{cases}$

Consider a problem with m equations and ℓ ($>m$) nonnegative variables. A **basic feasible solution (BFS)** is a solution for the problem that is obtained by setting $\ell - m$ variables equal to zero – **nonbasic variables (NBV)** – and solving the resulting system for the remaining m variables – **basic variables (BV)** – in case this system is feasible and determined, and with a nonnegative solution.

Each BFS is associated with a single corner point of the feasible region, named **corner point feasible solution (CPF)**. Each CPF is associated with at least one BFS.

Two BFSs are **adjacent** if all but one of their basic variables are the same (consequently, the respective sets of nonbasic variables also differ by one single variable).

Solving LP problems

Simplex algorithm (George Dantzig, 1947)

Let us consider an LP problem in the **standard form**[(1)] and with nonnegative right-hand-sides (RHS).

Step 1: write the problem in the augmented form

Step 2: determine a BFS
set $k \leftarrow 1$

{**Iteration *k***}

Step 3: if the BFS is optimal then **stop**
otherwise, continue

Step 4: apply the "entering criterion" to select x_p, the new BV
apply the "leaving criterion"
if not possible then **stop** (unbounded OF)
otherwise, let x_r be the new NBV

Step 5: update the new Simplex tableau thus building the new BFS (exchange variables x_p and x_r)
set $k \leftarrow k+1$
go to **step 3**

Notes:

entering criterion – select among the variables with negative coefficient at the OF row, the one with the most negative coefficient (variable x_p).

leaving criterion – in other to guarantee the feasibility of the new BFS, calculate the minimum of the ratios of the RHS and the positive coefficients of the entering column (variable x_p). The variable associated with that minimum (x_r) will be nonbasic in the new solution.

(1) The Simplex algorithm can be adapted to any LP problem.